A General Regression Changepoint Test for Time Series Data

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Collaborators:
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Changepoints and Climate

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- Seasonality
- Trend

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- $\beta_0, \beta_1, \ldots$??

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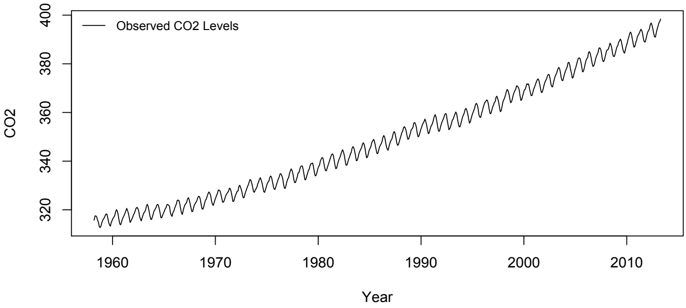

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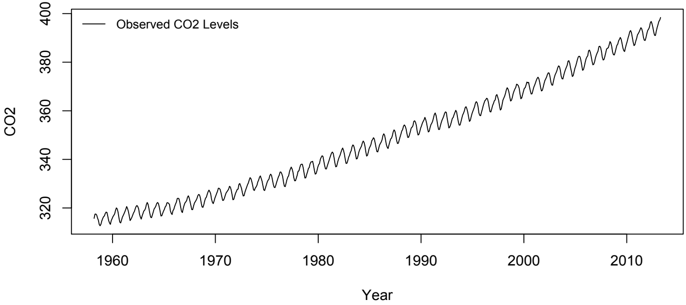

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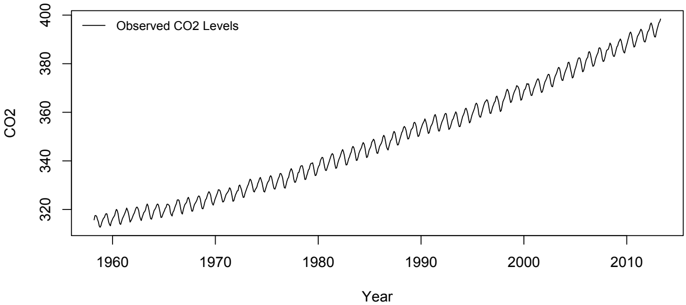

- Characteristics
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  - *Shifts in trend or in seasonality?*

Model the observed data y_t, for $1 \leq t \leq n$, as

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- The ϵ_t are *stationary* mean zero errors.
- $\widetilde{\alpha}$, $\widetilde{\boldsymbol{\beta}}$, and $\widetilde{\gamma}$ are vectors of regression coefficients.

Back to the Example

We assume the following model for the Mauna Loa CO_2 data:

$$\text{CO2}_t = \alpha_0 + \alpha_1 \left(\frac{t}{n}\right) + \alpha_2 \left(\frac{t}{n}\right)^2$$

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An equivalent representation is

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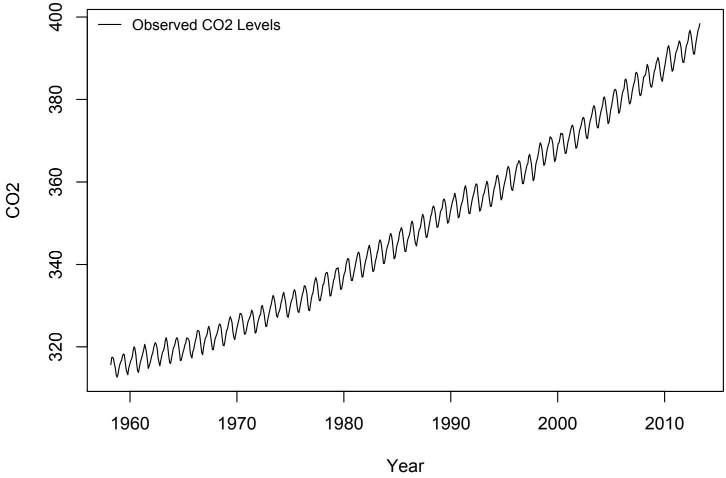
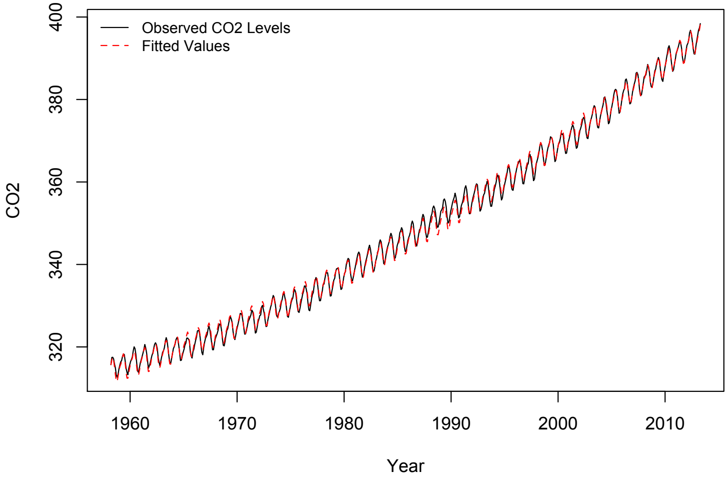
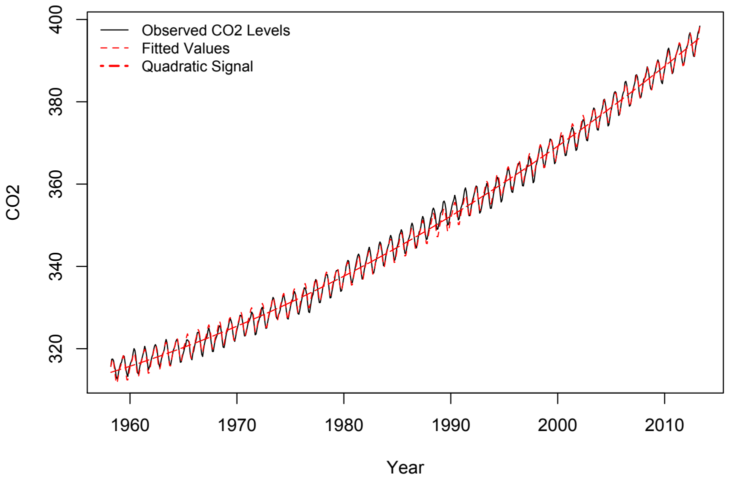


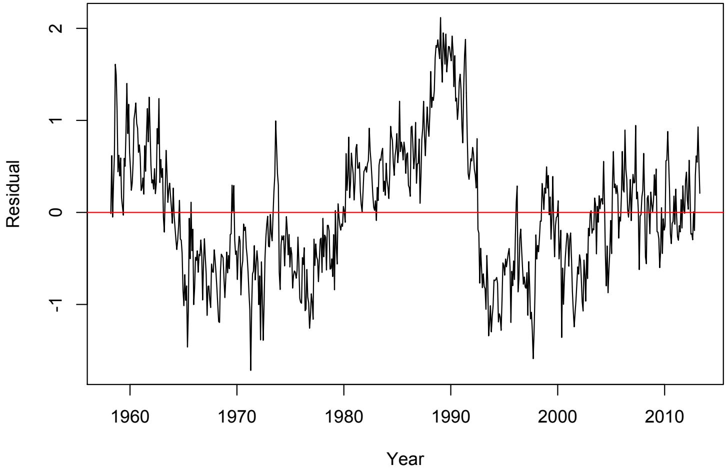
The Mauna Loa CO_2 data with fit:



The Mauna Loa CO_2 data with fit & trend:



The OLS residuals:



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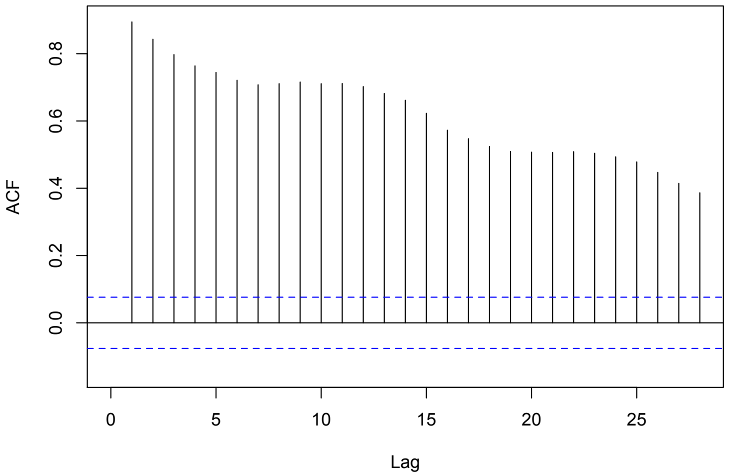
The ACF of the OLS residuals:



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Let $\widetilde{\boldsymbol{x}}_t = (\boldsymbol{x}_t', (\boldsymbol{x}_t^*)')'$, $\widetilde{\boldsymbol{s}}_t = (\boldsymbol{s}_t', (\boldsymbol{s}_t^*)')'$ and $\widetilde{\boldsymbol{x}}_t = (\boldsymbol{x}_t', (\boldsymbol{x}_t^*)')'$.

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We test $H_0 : \boldsymbol{\Delta} = \mathbf{0}$ vs. $H_1 : \boldsymbol{\Delta} \neq \mathbf{0}$.

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$$\widehat{F}_k = \widehat{\boldsymbol{\Delta}}_k' [\widehat{\textbf{Var}}(\widehat{\boldsymbol{\Delta}}_k)]^{-1} \widehat{\boldsymbol{\Delta}}_k = \widehat{\boldsymbol{\Delta}}_k' \boldsymbol{C}_k \widehat{\boldsymbol{\Delta}}_k / \hat{\tau}^2,$$

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which equivalent to the likelihood ratio and F-tests.

To detect a change at an *unknown* time, we consider

$$\widehat{F} = \max_{\ell \leq \frac{k}{n} \leq h} \widehat{F}_k$$

for truncation values ℓ and h that satisfy $0 < \ell < h < 1$.

Test Statistics for Changepoint Detection

Let $\hat{\mu}_t = \hat{\boldsymbol{\alpha}}'\widetilde{\boldsymbol{x}}_t + \hat{\boldsymbol{\beta}}'\widetilde{\boldsymbol{w}}_t + \hat{\boldsymbol{\gamma}}'\widetilde{\boldsymbol{s}}_t$.

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Let $\hat{\mu}_t = \hat{\boldsymbol{\alpha}}' \widetilde{\boldsymbol{x}}_t + \hat{\boldsymbol{\beta}}' \widetilde{\boldsymbol{w}}_t + \hat{\boldsymbol{\gamma}}' \widetilde{\boldsymbol{s}}_t$. Next, write

$$\hat{\epsilon}_t = y_t - \hat{\mu}_t.$$

The sequence $\{\hat{\epsilon}_t\}$ is referred to as the **OLS residuals**.

Test Statistics for Changepoint Detection

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We seek to express the $\widehat{F}_k$ statistic in terms of OLS residuals for two reasons:

- Helps when determining the sampling distribution (needed for critical values/p-values)
- Helps when extending the test to situations involving autocorrelation

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$\widehat{F}_{\mathrm{x},k}$, $\widehat{F}_{\mathrm{s},k}$, and $\widehat{F}_{\mathrm{v},k}$ are independent of one another.

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Note the following about the limit distribution

- $B_1(x)$ is an ugly Gaussian process
- $B_2(x) = \mathbf{B}_{p_s+p_v}(z)'\mathbf{B}_{p_s+p_v}(z)/[z(1-z)]$
  - $\mathbf{B}_d(z)$ is a d-dimensional set of independent Brownian bridges
- The limit distribution depends only on the form of $\boldsymbol{x}_t$ and $\boldsymbol{x}_t^*$ (and the dimensionality of $\boldsymbol{s}_t$ and $\boldsymbol{v}_t$).

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$$\hat{\tau}_B^2 = \frac{1}{n}\sum_{t=1}^{n} \hat{\epsilon}_t^2 + 2\sum_{s=1}^{q_n}\left(1 - \frac{s}{q_n + 1}\right)\frac{1}{n-s}\sum_{t=1}^{n-s} \hat{\epsilon}_t \hat{\epsilon}_{t+s},$$

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It is possible to construct versions of $\widehat{F}_{\mathrm{x},k}$, $\widehat{F}_{\mathrm{s},k}$, and $\widehat{F}_{\mathrm{v},k}$ that still use OLS residuals but encapsulate autocorrelation through a Bartlett-type variance estimator.

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- Convergence is slow.

ARMA Models

Can assume that ϵ_t obeys an ARMA($p_{\mathrm{ar}}, q_{\mathrm{ma}}$) model:

$$\epsilon_t - \phi_1 \epsilon_{t-1} - \cdots - \phi_{p_{\mathrm{ar}}} \epsilon_{t-p_{\mathrm{ar}}} = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_{q^*} Z_{t-q_{\mathrm{ma}}},$$

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- Faster convergence

Procedures Based on ARMA residuals

Let

$$\boldsymbol{R}_{\mathrm{x},k} = \sum_{t=1}^{k} \boldsymbol{x}_t \hat{Z}_t, \qquad \left(\text{OLS:} \quad \boldsymbol{N}_{\mathrm{x},k} = \sum_{t=1}^{k} \boldsymbol{x}_t \hat{\epsilon}_t\right)$$

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Thus, $\widehat{F}_{\mathrm{x},k}$ and $\widehat{L}_{\mathrm{x},k}$ are asymptotically equivalent.

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We can show that $\boldsymbol{N}_{\mathrm{s},k}$ and $\boldsymbol{R}_{\mathrm{s},k}$ have the same asymptotic distribution when scaled properly.

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Procedures Based on ARMA residuals

Let

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$$\boldsymbol{R}_{\mathrm{v},k} = \sum_{t=1}^{k} \boldsymbol{v}_t \hat{Z}_t - \frac{k}{n} \sum_{t=1}^{n} \boldsymbol{v}_t \hat{Z}_t, \qquad \left(\text{OLS:} \quad \boldsymbol{N}_{\mathrm{v},k} = \sum_{t=1}^{k} \boldsymbol{v}_t \hat{\epsilon}_t \right)$$

We can show that $\boldsymbol{N}_{\mathrm{v},k}$ and $\boldsymbol{R}_{\mathrm{v},k}$ have the same asymptotic distribution when scaled properly.

Also let

$$\widehat{L}_{\mathrm{v},k} = \frac{\boldsymbol{R}'_{\mathrm{v},k} \widehat{\boldsymbol{\Sigma}}_v^{-1} \boldsymbol{R}_{\mathrm{v},k}}{\hat{\sigma}^2 k(1 - \frac{k}{n})}, \qquad \left(\text{OLS:} \quad \widehat{F}_{\mathrm{v},k} = \boldsymbol{N}'_{\mathrm{v},k} \widehat{\boldsymbol{\tau}}_v^{-1} \boldsymbol{N}_{\mathrm{v},k} \right)$$

It follows that $\widehat{F}_{\mathrm{v},k}$ and $\widehat{L}_{\mathrm{v},k}$ have the same asymptotic distribution.

Procedures Based on ARMA residuals

An omnibus test based on ARMA residuals for a changepoint at time k is

$$\widehat{L}_k = \widehat{L}_{\mathrm{x},k} + \widehat{L}_{\mathrm{s},k} + \widehat{L}_{\mathrm{v},k}$$

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It follows that under H_0

$$\widehat{L} \xrightarrow{\mathcal{D}} \sup_{\ell < x < h} \{B_1(x) + B_2(x)\},$$

which is the limit process that was observed by the statistic $\widehat{F}$.

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The efficacy of the proposed methodology is studied via simulation.

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The error sequence $\{\epsilon_t\}$ is generated via an AR(1):

$$\epsilon_t = \phi_1 \epsilon_{t-1} + Z_t$$

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Four settings are examined (fix $n = 1000$ with $c = 500$ under H_1):

- Setting 1: **All** regression coefficients may change under H_1.
- Setting 2: Only coefficients governing trend may change.
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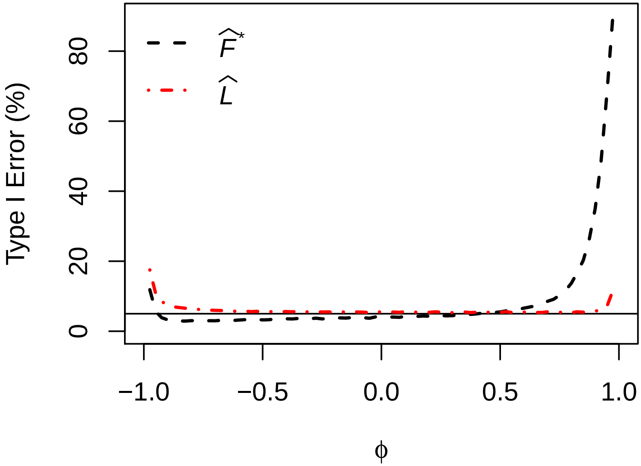

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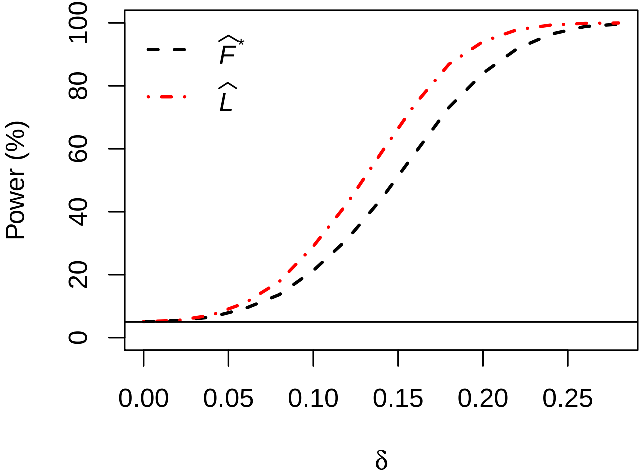
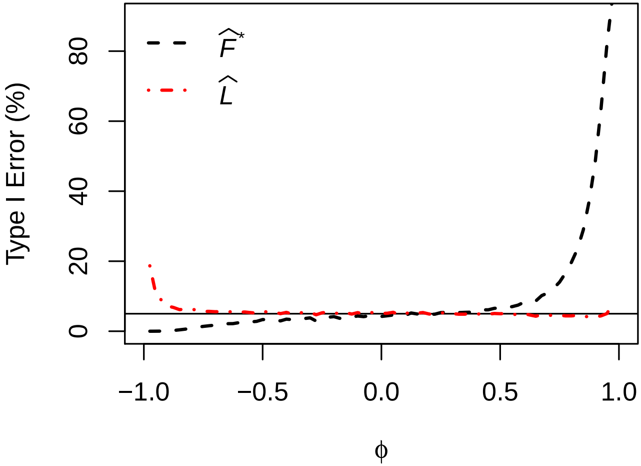


Setting 2: Only coefficients governing trend may change.



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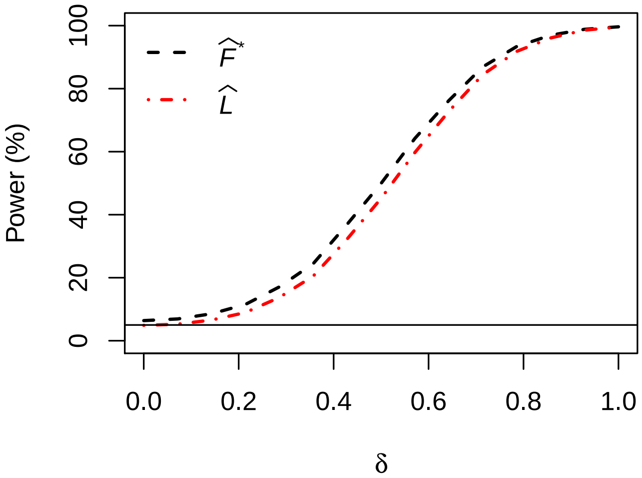

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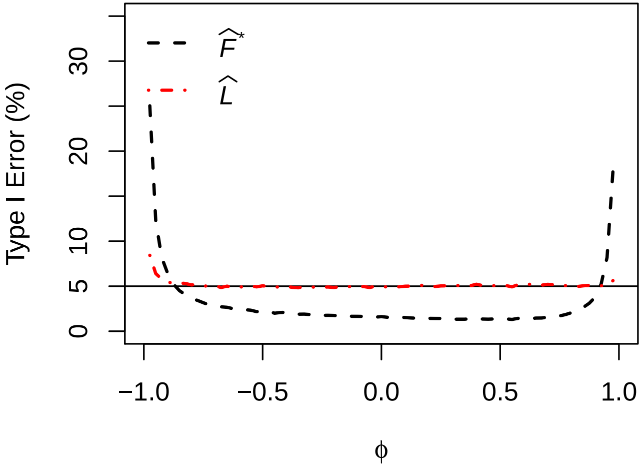

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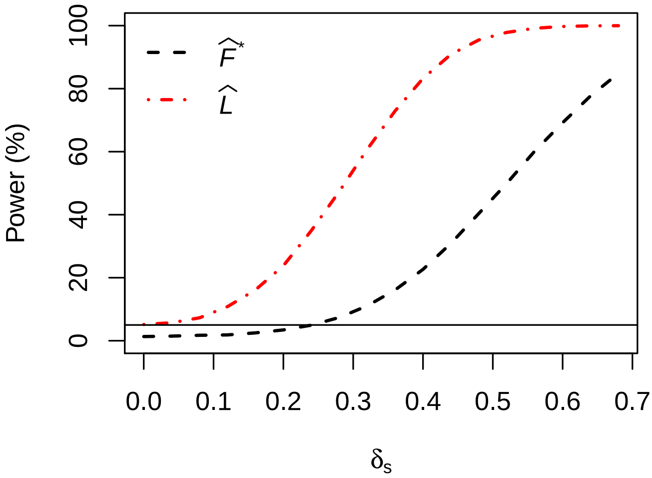

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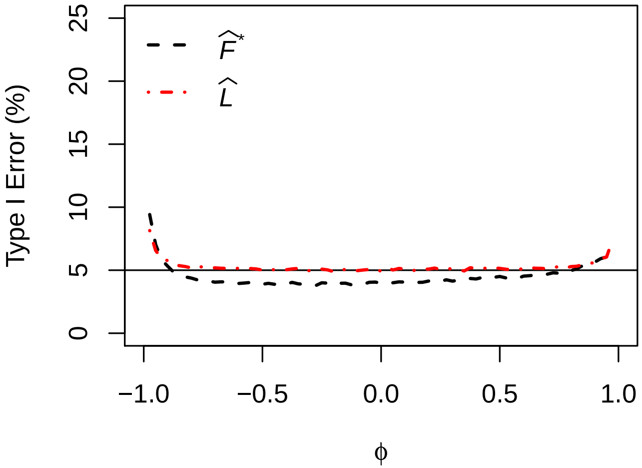

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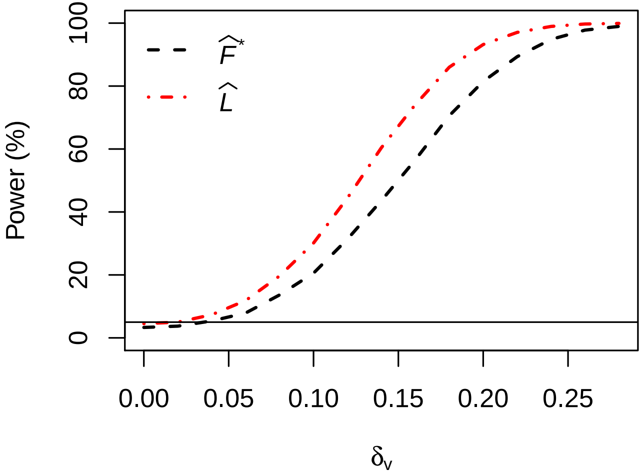

Fit an AR(p_{ar}) model to the OLS residuals of the CO_2 data.

Setting 2: Only coefficients governing trend change.

$\widehat{F}$				$\widehat{L}$			
q_n	$\hat{c}$	Stat.	p-value	p_{ar}	$\hat{c}$	Stat.	p-value
2	1991	155.1	0.0000	2	1991	24.1	0.0027
4	1991	97.5	0.0000	4	1991	20.3	0.0117
8	1991	57.5	0.0000	8	1991	18.9	0.0223
12	1991	41.2	0.0000	12	1991	20.5	0.0110
16	1991	32.3	0.0001	16	1991	16.3	0.0587
24	1991	23.3	0.0037	24	1991	17.7	0.0325

Table: Results of the tests for values of the autoregressive order (p_{ar}) and the lag cut-off for the Bartlett estimator (q_n).

Application to the CO_2 Data

Fit an AR(p_{ar}) model to the OLS residuals of the CO_2 data.

Setting 3: Only coefficients governing seasonality change.

$\widehat{F}$				$\widehat{L}$			
q_n	$\hat{c}$	Stat.	p-value	p_{ar}	$\hat{c}$	Stat.	p-value
2	1973	24.7	0.0520	2	1976	60.2	0.0000
4	1973	20.8	0.1706	4	1976	62.2	0.0000
8	1973	17.9	0.3610	8	1976	61.3	0.0000
12	2011	22.2	0.1132	12	1976	54.7	0.0000
16	2011	41.7	0.0001	16	1976	41.0	0.0001
24	2012	33.7	0.0018	24	1976	39.8	0.0002

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*Note: We assume a change in trend occurred in 1991

Application to the CO_2 Data

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Setting 4: Only coefficients governing covariates change.

$\widehat{F}$				$\widehat{L}$			
q_n	$\hat{c}$	Stat.	p-value	p_{ar}	$\hat{c}$	Stat.	p-value
2	1996	4.0	0.5016	2	2006	3.7	0.5529
4	1996	3.0	0.6911	4	2006	3.7	0.5462
8	1996	2.4	0.8193	8	2006	4.2	0.4696
12	1996	2.4	0.8360	12	2006	4.7	0.3860
16	1996	2.4	0.8332	16	2010	4.5	0.4191
24	1996	2.2	0.8681	24	2010	4.9	0.3509

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Fit an AR(p_{ar}) model to the OLS residuals of the CO_2 data.

Setting 1: All regression coefficients change under H_1

	$\widehat{F}$				$\widehat{L}$		
q_n	$\hat{c}$	Stat.	p-value	p_{ar}	$\hat{c}$	Stat.	p-value
2	1991	174.6	0.0000	2	1977	77.8	0.0000
4	1988	114.8	0.0000	4	1976	84.6	0.0000
8	1988	77.0	0.0000	8	1977	78.2	0.0000
12	1988	60.2	0.0000	12	1977	61.9	0.0000
16	1989	54.5	0.0000	16	1976	47.9	0.0004
24	1988	56.5	0.0000	24	1977	46.0	0.0006

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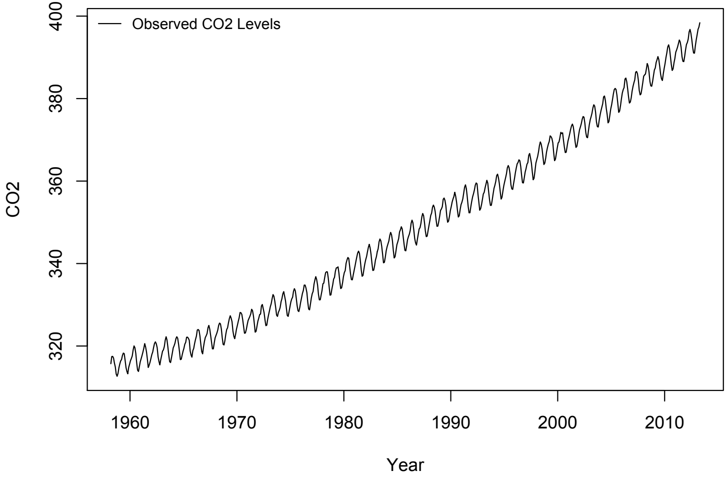
Application to the CO_2 Data

The Mauna Loa CO_2 data:



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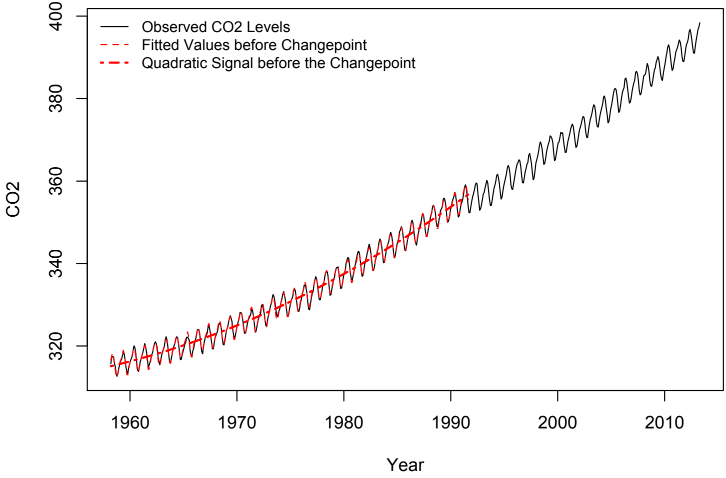

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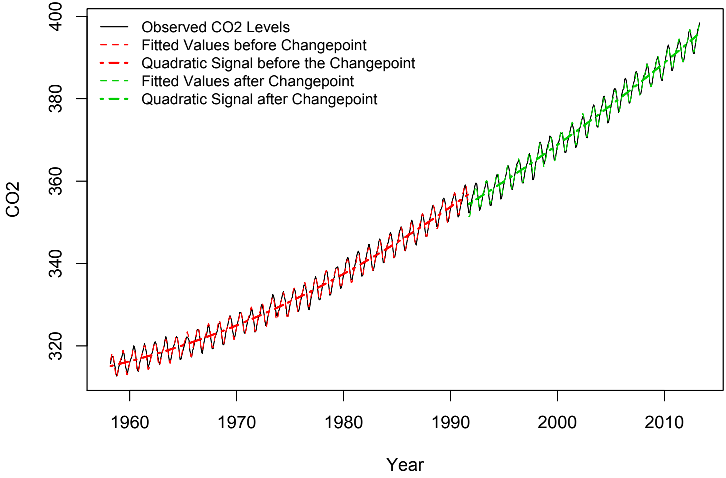
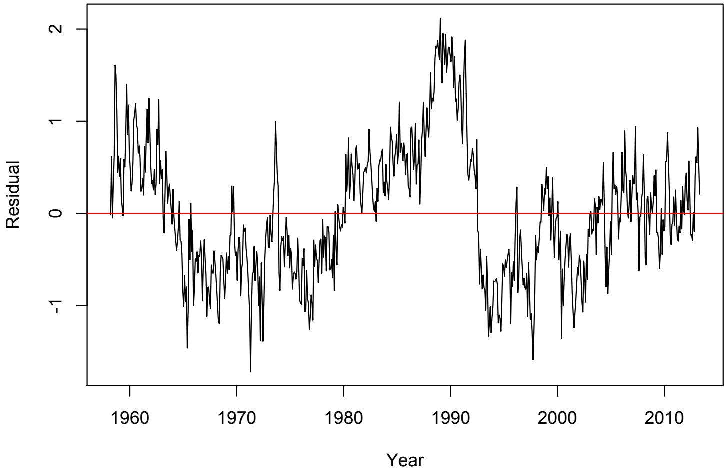


Application to the CO_2 Data

The OLS residuals without accounting for changepoint:



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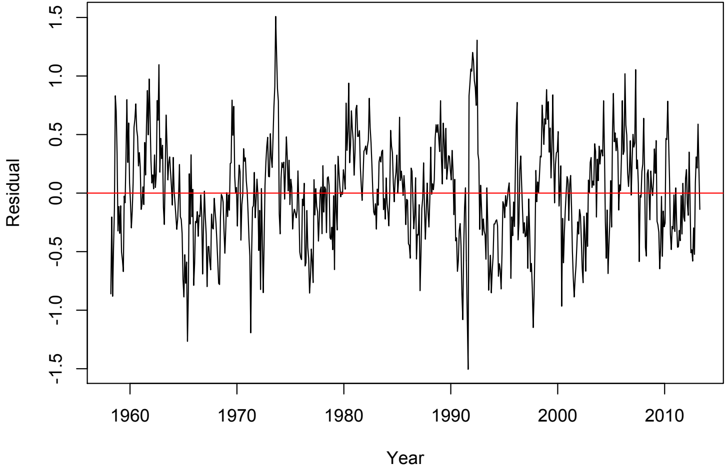
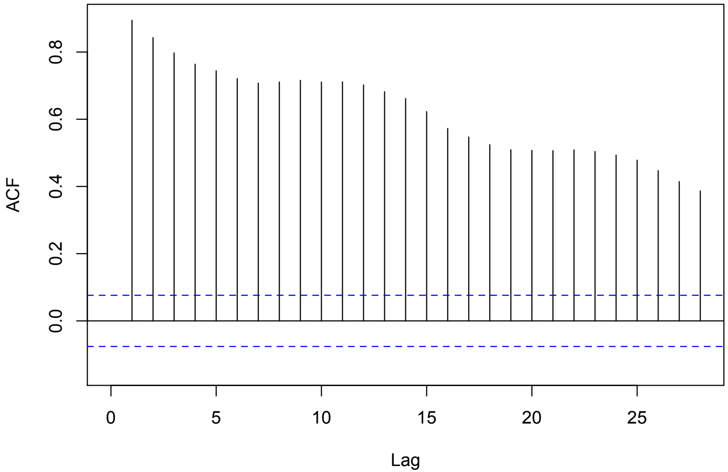
The OLS residuals while accounting for changepoint:



Application to the CO_2 Data

The ACF of the OLS residuals without accounting for changepoint:



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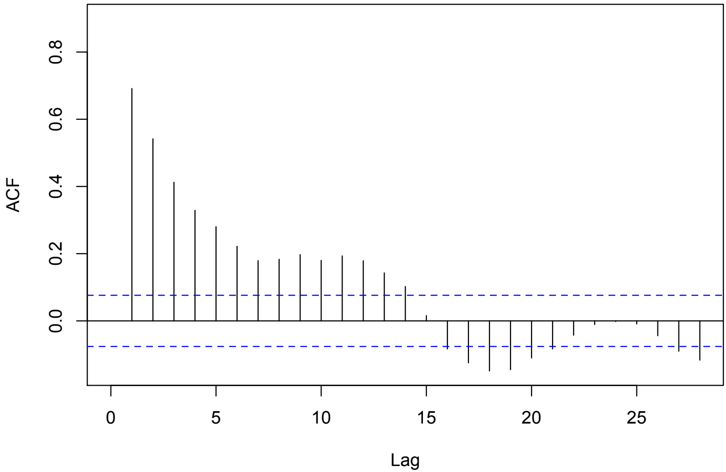

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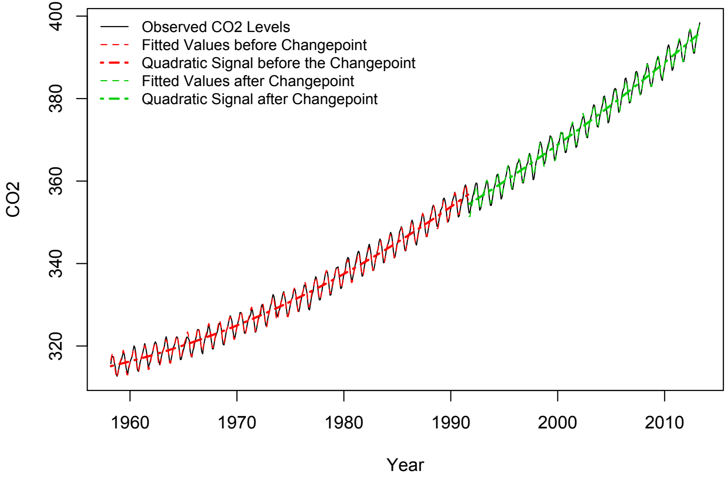
Mauna Loa is a volcano itself:

- Weak explosions
- Particles do not reach the stratosphere;
- No major effect on CO_2.

The Mauna Loa CO_2 data:



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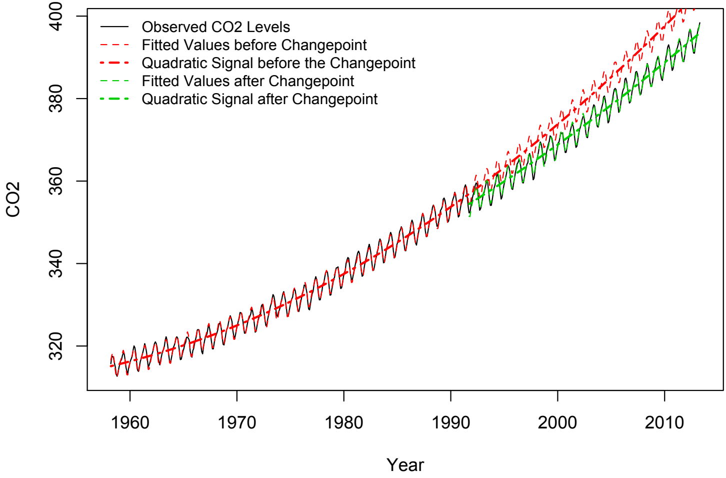
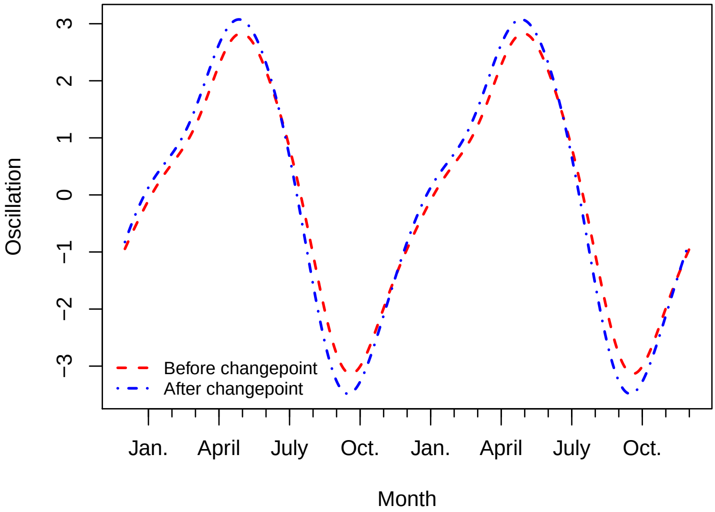


The 1976 changepoint in seasonality

The seasonal pattern before and after the 1976 changepoint:



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Why are the oscillations changing?

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Changes in trend and seasonality caused by separate environmental mechanisms

- Need to test for the different types of changes separately

References

Changes in trend only:

- Robbins, M. W., C. M. Gallagher and R. B. Lund (2016) "A General Regression Changepoint Test for Time Series Data." *Journal of the American Statistical Association*. Forthcoming in the June issue

Changes in all regression coefficients (trend, seasonality, covariates):

- Robbins, M. W. (2016) "A Fully Flexible Changepoint Test for Regression Models with Stationary Errors." Submitted.

`RegCpt`: A R package that implements the method

- In development

E-mail: `mrobbins@rand.org`